Exact Expressions for the Generalization Error in Statistical Federated Learning

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joint work with Samir Perlaza (Inria), Iñaki Esnaola (University of Sheffield) and H. Vincent Poor (Princeton University)

FedMalin General Assembly
November 27, 2025, Montpellier, France

Outline

Representation of Algorithms

P_Z Unknown Data Generating Distribution

$\triangle(\mathcal{X} \times \mathcal{Y})$

Representation of Algorithms

$z = ((x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n))$

Training Dataset

Algorithm
(SGD)

[1] W. Azizian, F. Lutzeler, J. Malick, and P. Mertikopoulos, "**What is the long-run distribution of stochastic gradient descent? A large deviations analysis**," in Proc. of the 41st International Conference on Machine Learning, July 2024, pp. 2168 – 2229.

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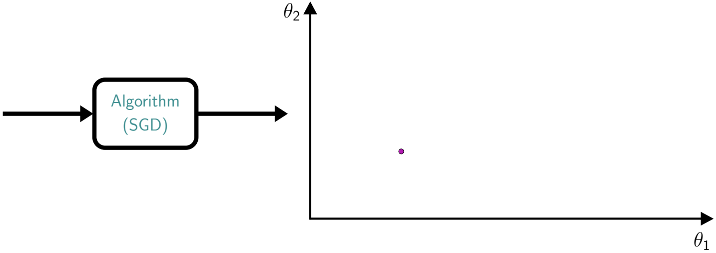

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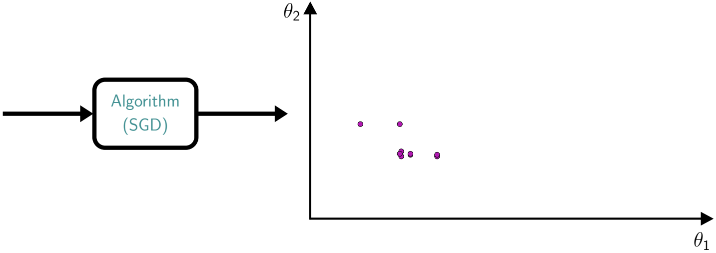

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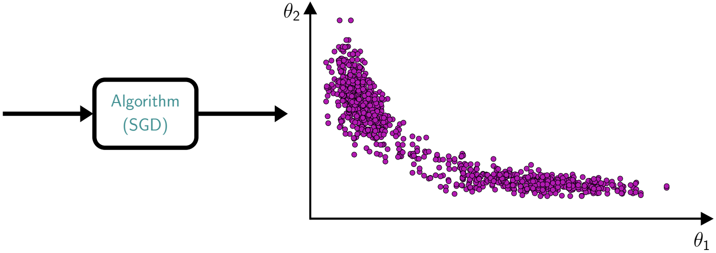

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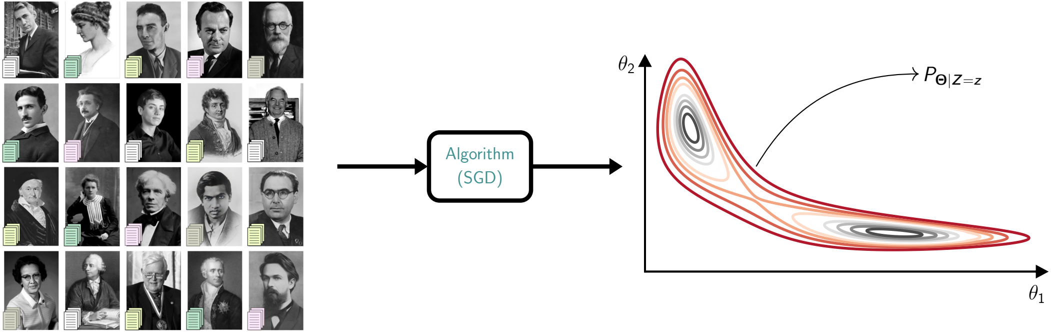

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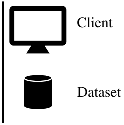
Federated Learning: Overview



For all $k \in \{1, 2, \ldots, K\}$, the training dataset of client k, $\boldsymbol{z}_k \in \mathcal{Z}_k^{n_k} \triangleq (\mathcal{X}_k \times \mathcal{Y}_k)^{n_k}$:

$$\boldsymbol{z}_k \triangleq ((x_{k,1}, y_{k,1}), (x_{k,2}, y_{k,2}), \ldots, (x_{k,n_k}, y_{k,n_k})),$$

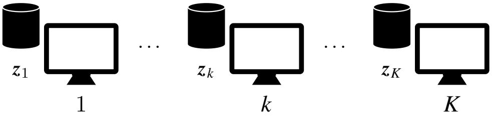
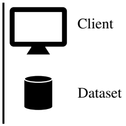


Figure: Federated learning scheme

Federated Learning: Overview



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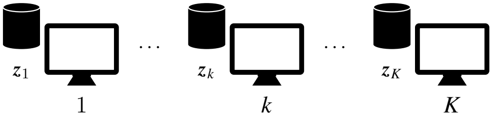
$$\boldsymbol{z}_k \triangleq \left((x_{k,1}, y_{k,1}), (x_{k,2}, y_{k,2}), \dots, (x_{k,n_k}, y_{k,n_k})\right),$$

The loss function of client k:

$$\ell_k : \mathcal{M}_k \times \mathcal{Z}_k \to [0, +\infty),$$



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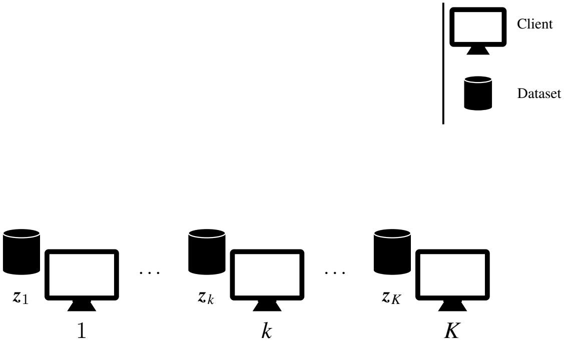

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For all $k \in \{1, 2, \ldots, K\}$, the training dataset of client k, $\boldsymbol{z}_k \in \mathcal{Z}_k^{n_k} \triangleq (\mathcal{X}_k \times \mathcal{Y}_k)^{n_k}$:

$$\boldsymbol{z}_k \triangleq \left((x_{k,1}, y_{k,1}), (x_{k,2}, y_{k,2}), \ldots, (x_{k,n_k}, y_{k,n_k})\right),$$

The loss function of client k:

$$\ell_k : \mathcal{M}_k \times \mathcal{Z}_k \to [0, +\infty),$$

The *empirical risk* induced by $\boldsymbol{\theta} \in \mathcal{M}_k$, with respect to the dataset $\boldsymbol{z}_k$:

$$\mathsf{L}_k : \begin{cases} \mathcal{Z}_k^{n_k} \times \mathcal{M}_k \longrightarrow [0, +\infty) \\ (\boldsymbol{z}_k, \boldsymbol{\theta}) \longmapsto \frac{1}{n_k} \sum_{i=1}^{n_k} \ell_k\left(\boldsymbol{\theta}, x_{k,i}, y_{k,i}\right), \end{cases}$$

[2] Samir M Perlaza, Gaetan Bisson, Iñaki Esnaola, Alain Jean-Marie, Stefano Rini. **Empirical Risk Minimization with Relative Entropy Regularization.** IEEE Transactions on Information Theory, 2024, 70 (7), pp.5122-5161.

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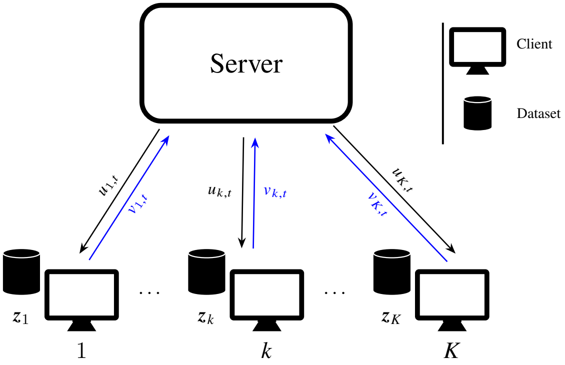

Figure: Federated learning scheme

For all $k \in \{1, 2, \dots, K\}$ and for $t \in \{0, \dots, m\}$,

- $\boldsymbol{z}_k \in \mathcal{Z}_k^{n_k}$;
- $u_{k,t} \in \mathcal{U}_k$;
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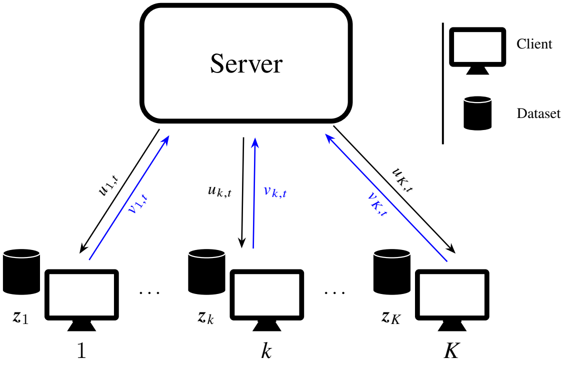

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Example (FedAvg [3])

Let $\mathcal{U}_k = \mathcal{V}_k = \mathcal{M}_k$ and $v_{k,0} = \varnothing$, $u_{k,0} = \boldsymbol{\theta}_0$ then, for $t \in \{1, \ldots, m\}$,

$$v_{k,t} = \nabla \mathrm{L}_k \left(\boldsymbol{z}_k, \, u_{k,t-1} \right),$$

$$u_{k,t} = u_{k,t-1} \; - \; \eta \sum_{i=1}^{K} \frac{n_i}{n_0} \, v_{i,t}.$$

[3] H. B. McMahan, E. Moore, D. Ramage, S. Hampson, and B. A. y Arcas, "**Communication-efficient learning of deep networks from decentralized data**", Proceedings of the 20th International Conference on Artificial Intelligence and Statistics (AISTATS), pp. 1273–1282, 2017

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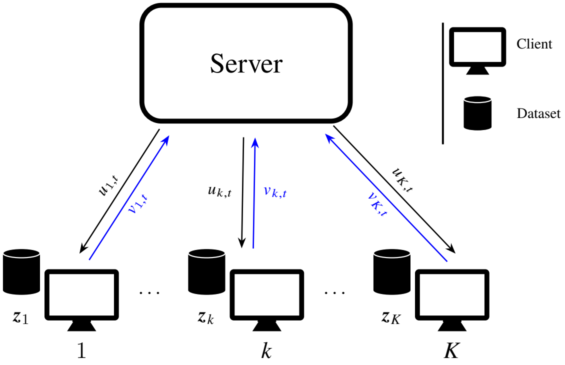

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- $u_{k,t} \in \mathcal{U}_k$;
- $v_{k,t} \in \mathcal{V}_k$;
- $\boldsymbol{\theta}_k \in \mathcal{M}_k$.

Communication constraints:

- **uplink** information transmission rate: $\overline{R}_k$;
- **downlink** information transmission rate: $\underline{R}_k$.

$$\boxed{|\mathcal{U}_k| \leqslant 2^{m\underline{R}_k} \quad \text{and} \quad |\mathcal{V}_k| \leqslant 2^{m\overline{R}_k}.}$$

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Federated Learning: Communication

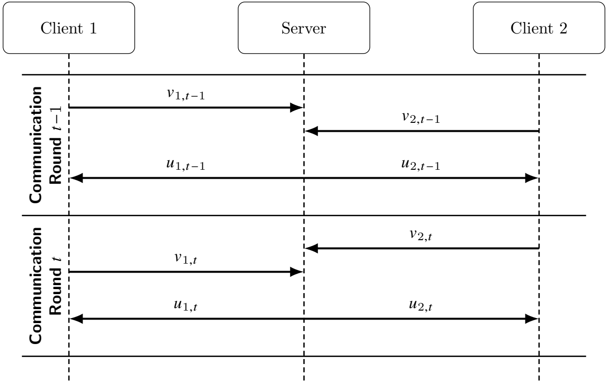


Figure: Two consecutive communications rounds, namely $t-1$ and t, with $t \in \{2, 3, \ldots, m\}$, in the case in which $K = 2$.

Federated Learning: Communication

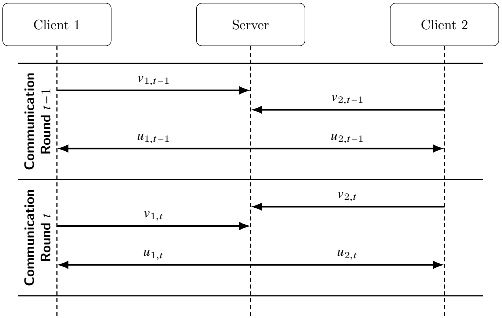


Figure: Two consecutive communications rounds, namely $t-1$ and t, with $t \in \{2, 3, \ldots, m\}$, in the case in which $K = 2$.

For all $k \in \{1, 2, \ldots, K\}$ and for $t \in \{0, \ldots, m\}$,

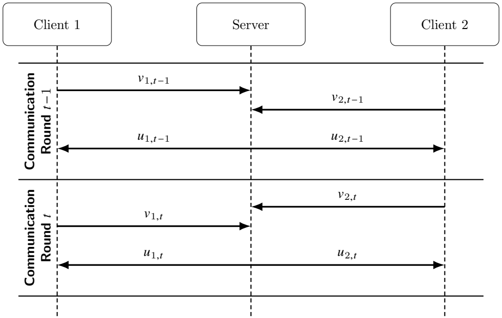
- all messages sent to the server during round t:

$$\boldsymbol{v}^{(t)} = \left(v_{1,t}, v_{2,t}, \ldots, v_{K,t}\right)^{\top} \in \mathcal{V}_1 \times \mathcal{V}_2 \times \ldots \times \mathcal{V}_K;$$

- all messages sent by client k during all previous rounds, including round t:

$$\boldsymbol{v}_k^{(t)} = \left(v_{k,1}, v_{k,2}, \ldots, v_{k,t}\right)^{\top} \in \mathcal{V}_k^t.$$

Federated Learning: Communication



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- all messages sent by client k during all previous rounds, including round t:

$$\boldsymbol{v}_k^{(t)} = (v_{k,1}, v_{k,2}, \ldots, v_{k,t})^{\top} \in \mathcal{V}_k^t.$$

Let $\underline{\boldsymbol{v}}^{(t)} \in (\mathcal{V}_1 \times \mathcal{V}_2 \times \ldots \times \mathcal{V}_K)^t$ be such that,

$$\underline{\boldsymbol{v}}^{(t)} \triangleq \begin{bmatrix} v_{1,1} & v_{1,2} & \cdots & v_{1,t} \\ v_{2,1} & v_{2,2} & \cdots & v_{2,t} \\ \vdots & \vdots & \ddots & \vdots \\ v_{K,1} & v_{K,2} & \cdots & v_{K,t} \end{bmatrix}$$
$$= \left(\boldsymbol{v}_1^{(t)} \; \boldsymbol{v}_2^{(t)} \; \cdots \; \boldsymbol{v}_K^{(t)} \right)^{\top}$$
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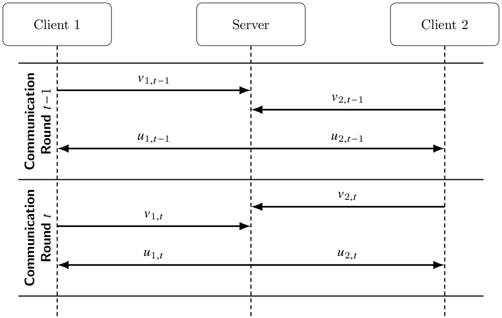

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For all $k \in \{1, 2, \ldots, K\}$ and for $t \in \{0, \ldots, m\}$,

- all messages sent by the server during round t:

$$\boldsymbol{u}^{(t)} = (u_{1,t}, u_{2,t}, \ldots, u_{K,t})^{\top} \in \mathcal{U}_1 \times \mathcal{U}_2 \times \ldots \times \mathcal{U}_K;$$

- all messages sent by the server to client k during all previous rounds, including round t:

$$\boldsymbol{u}_k^{(t)} = (u_{k,1}, u_{k,2}, \ldots, u_{k,t})^{\top} \in \mathcal{U}_k^t.$$

Let $\underline{\boldsymbol{u}}^{(t)} \in (\mathcal{U}_1 \times \mathcal{U}_2 \times \ldots \times \mathcal{U}_K)^t$ be such that,

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Federated Learning: Statistical Modelling

$P_{\underline{\mathbf{\Theta}},\, \underline{\boldsymbol{U}}^{(m)}, \underline{\boldsymbol{V}}^{(m)}, \boldsymbol{Z}_0} \in \triangle\left(\mathcal{M}_0 \times \mathcal{U}_0^m \times \mathcal{V}_0^m \times \mathcal{Z}_0\right)$, with

$$
\begin{aligned}
\mathcal{M}_0 &\triangleq \mathcal{M}_1 \times \mathcal{M}_2 \times \ldots \times \mathcal{M}_K \subset \mathbb{R}^{d \times K}; \\
\mathcal{U}_0 &\triangleq \mathcal{U}_1 \times \mathcal{U}_2 \times \ldots \times \mathcal{U}_K; \\
\mathcal{V}_0 &\triangleq \mathcal{V}_1 \times \mathcal{V}_2 \times \ldots \times \mathcal{V}_K; \\
\mathcal{Z}_0 &\triangleq \mathcal{Z}_1^{n_1} \times \mathcal{Z}_2^{n_2} \times \ldots \times \mathcal{Z}_K^{n_K}.
\end{aligned}
$$

Federated Learning: Statistical Modelling

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$$\begin{aligned} \mathcal{M}_0 &\triangleq \mathcal{M}_1 \times \mathcal{M}_2 \times \ldots \times \mathcal{M}_K \subset \mathbb{R}^{d \times K}; \\ \mathcal{U}_0 &\triangleq \mathcal{U}_1 \times \mathcal{U}_2 \times \ldots \times \mathcal{U}_K; \\ \mathcal{V}_0 &\triangleq \mathcal{V}_1 \times \mathcal{V}_2 \times \ldots \times \mathcal{V}_K; \\ \mathcal{Z}_0 &\triangleq \mathcal{Z}_1^{n_1} \times \mathcal{Z}_2^{n_2} \times \ldots \times \mathcal{Z}_K^{n_K}. \end{aligned}$$

$$\boxed{P_{\underline{\mathbf{\Theta}},\, \underline{\boldsymbol{U}}^{(m)},\, \underline{\boldsymbol{V}}^{(m)},\, \boldsymbol{Z}_0} = P_{\underline{\mathbf{\Theta}},\, \underline{\boldsymbol{U}}^{(m)},\, \underline{\boldsymbol{V}}^{(m)} | \boldsymbol{Z}_0} P_{\boldsymbol{Z}_0}}$$

with,

$$\begin{aligned} P_{\underline{\mathbf{\Theta}},\, \underline{\boldsymbol{U}}^{(m)},\, \underline{\boldsymbol{V}}^{(m)} | \boldsymbol{Z}_0} &\in \triangle \left(\mathcal{M}_0 \times \mathcal{U}_0^m \times \mathcal{V}_0^m \mid \mathcal{Z}_0 \right); \text{ and} \\ P_{\boldsymbol{Z}_0} &\in \triangle \left(\mathcal{Z}_0 \right). \end{aligned}$$

Federated Learning: Statistical Modelling

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$$

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Federated Learning: Statistical Modelling

$$P_{\underline{\boldsymbol{\Theta}}, \underline{\boldsymbol{U}}^{(m)}, \underline{\boldsymbol{V}}^{(m)}, \boldsymbol{Z}_0} \in \triangle\left(\mathcal{M}_0 \times \mathcal{U}_0^m \times \mathcal{V}_0^m \times \mathcal{Z}_0\right), \text{ with}$$

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with,

$$\begin{aligned}
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\end{aligned}$$

The Machine Learning Algorithm

Assumption

The model chosen by client k, with $k \in \{1, 2, \ldots, K\}$, is obtained by sampling a probability measure exclusively conditioned on

- *The dataset $\boldsymbol{z}_k$;*
- *The m messages received from the server, $\boldsymbol{u}_k^{(m)}$; and*
- *The m messages sent to the server, $\boldsymbol{v}_k^{(m)}$.*

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- *The dataset $\boldsymbol{z}_k$;*
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- *The m messages sent to the server, $\boldsymbol{v}_k^{(m)}$.*

The *machine learning algorithm* used by client k:

$$\boxed{P_{\Theta_k | \boldsymbol{U}_k^{(m)}, \boldsymbol{V}_k^{(m)}, \boldsymbol{Z}_k} \in \triangle\left(\mathcal{M}_k | \mathcal{U}_k^m \times \mathcal{V}_k^m \times \mathcal{Z}_k^{n_k}\right).}$$

The Messenger of Client k

Assumption

The message sent by client k to the server at communication round t, with $k \in \{1, 2, \ldots, K\}$ and $t \in \{1, 2, \ldots, m\}$, is obtained by sampling a probability measure exclusively conditioned on

- *The dataset $\boldsymbol{z}_k$;*
- *The $t-1$ messages previously received from the server, $\boldsymbol{u}_k^{(t-1)}$; and*
- *The $t-1$ messages previously sent to the server, $\boldsymbol{v}_k^{(t-1)}$.*

The Messenger of Client k

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- *The $t-1$ messages previously received from the server, $\boldsymbol{u}_k^{(t-1)}$; and*
- *The $t-1$ messages previously sent to the server, $\boldsymbol{v}_k^{(t-1)}$.*

The *messenger* of client k during communication round t:

$$\boxed{P_{V_{k,t}|\boldsymbol{U}_k^{(t-1)}, \boldsymbol{V}_k^{(t-1)}, \boldsymbol{Z}_k} \in \triangle\left(\mathcal{V}_k | \mathcal{U}_k^{t-1} \times \mathcal{V}_k^{t-1} \times \mathcal{Z}_k^{n_k}\right).}$$

The Server

Assumption

The messages sent by the server at communication round t to all clients, $\boldsymbol{u}^{(t)}$, with $t \in \{1, 2, \ldots, m\}$, are obtained by sampling a probability measure exclusively conditioned on

- *The t messages received from each client, $\underline{\boldsymbol{v}}^{(t)}$; and*
- *The $t - 1$ messages previously sent to the clients, $\underline{\boldsymbol{u}}^{(t-1)}$.*

The Server

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The messages sent by the server at communication round t to all clients, $\boldsymbol{u}^{(t)}$, with $t \in \{1, 2, \ldots, m\}$, are obtained by sampling a probability measure exclusively conditioned on

- *The t messages received from each client, $\underline{\boldsymbol{v}}^{(t)}$; and*
- *The $t-1$ messages previously sent to the clients, $\underline{\boldsymbol{u}}^{(t-1)}$.*

The *server* during communication round t:

$$\boxed{P_{\boldsymbol{U}^{(t)}|\underline{\boldsymbol{U}}^{(t-1)},\underline{\boldsymbol{V}}^{(t)}} \in \triangle\left(\mathcal{U}_0|\mathcal{U}_0^{t-1} \times \mathcal{V}_0^t\right).}$$

Statistical Federated Learning

$$P_{\underline{\boldsymbol{\Theta}}, \underline{\boldsymbol{U}}^{(m)}, \underline{\boldsymbol{V}}^{(m)} | \boldsymbol{Z}_0} = \left(\prod_{k=1}^{K} P_{\boldsymbol{\Theta}_k | \boldsymbol{U}_k^{(m)}, \boldsymbol{V}_k^{(m)}, \boldsymbol{Z}_k} \right) \prod_{t=1}^{m} \left(P_{\boldsymbol{U}^{(t)} | \underline{\boldsymbol{U}}^{(t-1)}, \underline{\boldsymbol{V}}^{(t-1)}} \prod_{j=1}^{K} P_{V_{j,t} | \boldsymbol{U}_j^{(t-1)}, \boldsymbol{V}_j^{(t-1)}, \boldsymbol{Z}_j} \right).$$

Learning Algorithm

Statistical Federated Learning

$$P_{\underline{\boldsymbol{\Theta}}, \underline{\boldsymbol{U}}^{(m)}, \underline{\boldsymbol{V}}^{(m)} | \boldsymbol{Z}_0} = \left(\prod_{k=1}^{K} P_{\boldsymbol{\Theta}_k | \boldsymbol{U}_k^{(m)}, \boldsymbol{V}_k^{(m)}, \boldsymbol{Z}_k} \right) \prod_{t=1}^{m} \left(P_{\boldsymbol{U}^{(t)} | \underline{\boldsymbol{U}}^{(t-1)}, \underline{\boldsymbol{V}}^{(t-1)}} \prod_{j=1}^{K} P_{V_{j,t} | \boldsymbol{U}_j^{(t-1)}, \boldsymbol{V}_j^{(t-1)}, \boldsymbol{Z}_j} \right).$$

Learning Algorithm

Server

Statistical Federated Learning

$$P_{\underline{\boldsymbol{\Theta}},\underline{\boldsymbol{U}}^{(m)},\underline{\boldsymbol{V}}^{(m)}|\boldsymbol{Z}_0} = \left(\prod_{k=1}^{K} P_{\boldsymbol{\Theta}_k|\boldsymbol{U}_k^{(m)},\boldsymbol{V}_k^{(m)},\boldsymbol{Z}_k}\right) \prod_{t=1}^{m} \left(P_{\boldsymbol{U}^{(t)}|\underline{\boldsymbol{U}}^{(t-1)},\underline{\boldsymbol{V}}^{(t-1)}} \prod_{j=1}^{K} P_{V_{j,t}|\boldsymbol{U}_j^{(t-1)},\boldsymbol{V}_j^{(t-1)},\boldsymbol{Z}_j}\right).$$

Learning Algorithm

Server

Messenger

Example

Example (FedAvg)

$$P_{U_{k,t}|U_{k,t-1}=u_{k,t-1},\, \boldsymbol{V}^{(t)}=\boldsymbol{v}^{(t)}}\left(\{u_{k,t}\}\right) = \mathbb{1}_{\left\{u_{k,t}=u_{k,t-1}-\eta\sum_{i=1}^{K}\frac{n_i}{n_0}v_{i,t}\right\}} \in \triangle\left(\mathcal{U}_k\right),$$

$$P_{V_{k,t}|U_{k,t-1}=u_{k,t-1},\, \boldsymbol{Z}_k=\boldsymbol{z}_k}\left(\{v_{k,t}\}\right) = \mathbb{1}_{\left\{v_{k,t}=\nabla \mathsf{L}_k\left(\boldsymbol{z}_k,\, u_{k,t-1}\right)\right\}} \in \triangle\left(\mathcal{V}_k\right),$$

$$P_{\boldsymbol{\Theta}_k|U_{k,t}=u_{k,t},\, \boldsymbol{Z}_k=\boldsymbol{z}_k}\left(\{\boldsymbol{\theta}_k\}\right) = \mathbb{1}_{\left\{\boldsymbol{\theta}_k=u_{k,t}\right\}} \in \triangle\left(\mathcal{M}_k\right).$$

Example

Example (FedAvg)

$$P_{U_{k,t}|U_{k,t-1}=u_{k,t-1},\,\boldsymbol{V}^{(t)}=\boldsymbol{v}^{(t)}}\left(\{u_{k,t}\}\right) = \mathbb{1}_{\left\{u_{k,t}=u_{k,t-1}-\eta\sum_{i=1}^{K}\frac{n_i}{n_0}v_{i,t}\right\}} \in \triangle\left(\mathcal{U}_k\right),$$

$$P_{V_{k,t}|U_{k,t-1}=u_{k,t-1},\,\boldsymbol{Z}_k=\boldsymbol{z}_k}\left(\{v_{k,t}\}\right) = \mathbb{1}_{\left\{v_{k,t}=\nabla\mathsf{L}_k\left(\boldsymbol{z}_k,\,u_{k,t-1}\right)\right\}} \in \triangle\left(\mathcal{V}_k\right),$$

$$P_{\boldsymbol{\Theta}_k|U_{k,t}=u_{k,t},\,\boldsymbol{Z}_k=\boldsymbol{z}_k}\left(\{\boldsymbol{\theta}_k\}\right) = \mathbb{1}_{\left\{\boldsymbol{\theta}_k=u_{k,t}\right\}} \in \triangle\left(\mathcal{M}_k\right).$$

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Generalization Error: Meta Federated Learning Algorithm

Lemma

The marginal probability measure in $\triangle(\mathcal{M}_k \mid \mathcal{Z}_0)$ *of the probability measure* $P_{\underline{\boldsymbol{\Theta}}, \underline{\boldsymbol{U}}^{(m)}, \underline{\boldsymbol{V}}^{(m)} | \boldsymbol{Z}_0}$, *denoted* $P^{(m)}_{\boldsymbol{\Theta}_k | \boldsymbol{Z}_0} \in \triangle(\mathcal{M}_k \mid \mathcal{Z}_0)$, *is such that for all measurable sets* $\mathcal{A} \subseteq \mathcal{M}_k$ *and for all* $\boldsymbol{z}_0 = (\boldsymbol{z}_1, \boldsymbol{z}_2, \ldots, \boldsymbol{z}_K) \in \mathcal{Z}_0$,

$$P^{(m)}_{\boldsymbol{\Theta}_k | \boldsymbol{Z}_0 = \boldsymbol{z}_0}(\mathcal{A}) = \sum_{\left(\underline{\boldsymbol{u}}^{(m)}, \underline{\boldsymbol{v}}^{(m)}\right) \in \mathcal{U}_0^m \times \mathcal{V}_0^m} a_{\underline{\boldsymbol{u}}^{(m)}, \underline{\boldsymbol{v}}^{(m)}, \boldsymbol{z}_0} P_{\boldsymbol{\Theta}_k | \boldsymbol{U}_k^{(m)} = \boldsymbol{u}_k^{(m)},\, \boldsymbol{V}_k^{(m)} = \boldsymbol{v}_k^{(m)},\, \boldsymbol{Z}_k = \boldsymbol{z}_k}(\mathcal{A}).$$

Moreover, $a_{\underline{\boldsymbol{u}}^{(m)}, \underline{\boldsymbol{v}}^{(m)}, \boldsymbol{z}_0} \geqslant 0$ *and* $\sum_{\left(\underline{\boldsymbol{u}}^{(m)}, \underline{\boldsymbol{v}}^{(m)}\right) \in \mathcal{U}_0^m \times \mathcal{V}_0^m} a_{\underline{\boldsymbol{u}}^{(m)}, \underline{\boldsymbol{v}}^{(m)}, \boldsymbol{z}_0} = 1.$

$$a_{\underline{\boldsymbol{u}}^{(m)}, \underline{\boldsymbol{v}}^{(m)}, \boldsymbol{z}_0} \triangleq \prod_{t=1}^{m} P_{\boldsymbol{U}^{(t)} | \underline{\boldsymbol{U}}^{(t-1)} = \underline{\boldsymbol{u}}^{(t-1)}, \underline{\boldsymbol{V}}^{(t)} = \underline{\boldsymbol{v}}^{(t)}}\left(\left\{\boldsymbol{u}^{(t)}\right\}\right) \prod_{j=1}^{K} P_{V_{j,t} | \boldsymbol{U}_j^{(t-1)} = \boldsymbol{u}_j^{(t-1)}, \boldsymbol{V}_j^{(t-1)} = \boldsymbol{v}_j^{(t-1)}, \boldsymbol{Z}_j = \boldsymbol{z}_j}\left(\left\{v_{j,t}\right\}\right).$$

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Generalization Error: Definition

For some fixed dataset $\boldsymbol{z}_k \in \mathcal{Z}_k^{n_k}$,

$$\mathsf{R}_{k,\boldsymbol{z}_k} : \begin{cases} \triangle\left(\mathcal{M}_k\right) \longrightarrow [0, +\infty) \\ \qquad P \longmapsto \int \mathsf{L}_k\left(\boldsymbol{z}_k, \boldsymbol{\theta}\right) \mathrm{d}P\left(\boldsymbol{\theta}\right) \end{cases}.$$

Generalization Error: Definition

For some fixed dataset $\boldsymbol{z}_k \in \mathcal{Z}_k^{n_k}$,

$$\mathsf{R}_{k,\boldsymbol{z}_k} : \begin{cases} \triangle\left(\mathcal{M}_k\right) \longrightarrow [0,+\infty) \\ \qquad\quad P \longmapsto \int \mathsf{L}_k\left(\boldsymbol{z}_k, \boldsymbol{\theta}\right) \mathrm{d}P\left(\boldsymbol{\theta}\right) \end{cases}.$$

Definition (Generalization Error)

In the federated learning system described by the measure $P_{\underline{\boldsymbol{\Theta}}, \underline{\boldsymbol{U}}^{(m)}, \underline{\boldsymbol{V}}^{(m)} | \boldsymbol{Z}_0}$, under the assumption that datasets $\boldsymbol{z}_0 = (\boldsymbol{z}_1, \boldsymbol{z}_2, \ldots, \boldsymbol{z}_K) \in \mathcal{Z}_0$ are obtained by sampling a probability measure $P_{\boldsymbol{Z}_0} \in \triangle\left(\mathcal{Z}_0\right)$, the generalization error at client k is

$$\mathbf{G}_k\left(P_{\underline{\boldsymbol{\Theta}}, \underline{\boldsymbol{U}}^{(m)}, \underline{\boldsymbol{V}}^{(m)} | \boldsymbol{Z}_0}; P_{\boldsymbol{Z}_0}\right) \triangleq \iint \left(\mathsf{R}_{k,\hat{\boldsymbol{z}}_k}\left(P^{(m)}_{\boldsymbol{\Theta}_k | \boldsymbol{Z}_0 = \boldsymbol{z}_0}\right) - \mathsf{R}_{k,\boldsymbol{z}_k}\left(P^{(m)}_{\boldsymbol{\Theta}_k | \boldsymbol{Z}_0 = \boldsymbol{z}_0}\right) \right) \mathrm{d}P_{\boldsymbol{Z}_k}(\hat{\boldsymbol{z}}_k) \mathrm{d}P_{\boldsymbol{Z}_0}(\boldsymbol{z}_0).$$

[4] Samir M. Perlaza and Xinying Zou. "**The Generalization Error of Machine Learning Algorithms**". November, 2024.

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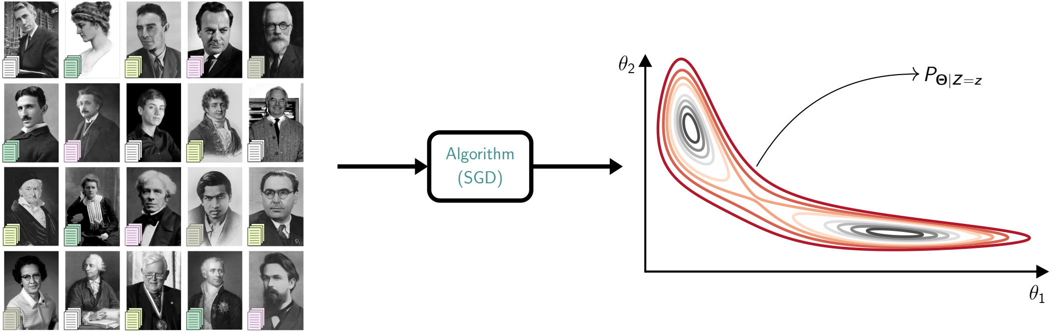
Representation of Algorithms



$z = ((x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n))$

Training Dataset

[1] W. Azizian, F. Lutzeler, J. Malick, and P. Mertikopoulos, "**What is the long-run distribution of stochastic gradient descent? A large deviations analysis**," in Proc. of the 41st International Conference on Machine Learning, July 2024, pp. 2168 – 2229.

Optimization problems

Let $\lambda_k > 0$,

$$\boxed{\min_{P \in \triangle_{Q_{\mathbf{\Theta}_k}}(\mathcal{M}_k)} \mathsf{R}_{k,\boldsymbol{z}_k}(P) + \lambda_k \,\mathsf{D}\left(P \| Q_{\mathbf{\Theta}_k}\right)}$$

Let $\lambda_k < 0$,

$$\boxed{\max_{P \in \triangle_{Q_{\mathbf{\Theta}_k}}(\mathcal{M}_k)} \mathsf{R}_{k,\boldsymbol{z}_k}(P) + \lambda_k \,\mathsf{D}\left(P \| Q_{\mathbf{\Theta}_k}\right)}$$

[5] Samir M. Perlaza and Gaetan Bisson. "**Variations on the expectation due to changes in the probability measure**". Entropy, 27(8:865):1–20, Aug. 2025.

Generalization Error: Gibbs Measure

Definition

Given the function L_k in (1), with $k \in \{1, 2, \ldots, K\}$; a σ-finite measure $Q_{\Theta_k} \in \triangle(\mathcal{M}_k)$; and a $\lambda_k \in \mathbb{R}\backslash\{0\}$, the probability measure $P^{(Q_{\Theta_k}, \lambda_k)}_{\Theta_k | \boldsymbol{Z}_k} \in \triangle(\mathcal{M}_k | \mathcal{Z}_k^{n_k})$ is said to be an $(\mathsf{L}_k, Q_{\Theta_k}, \lambda_k)$-Gibbs conditional probability measure if for all $\boldsymbol{z}_k \in \mathcal{S}_k$, $\mathsf{K}_{k, Q_{\Theta_k}, \boldsymbol{z}_k}\left(-\frac{1}{\lambda_k}\right) < +\infty$; for some set $\mathcal{S}_k \subseteq \mathcal{Z}_k^{n_k}$; and for all $(\boldsymbol{z}_k, \boldsymbol{\theta}_k) \in \mathcal{S}_k \times \operatorname{supp} Q_{\Theta_k}$,

$$\frac{\mathrm{d}P^{(Q_{\Theta_k}, \lambda_k)}_{\Theta_k | \boldsymbol{Z}_k = \boldsymbol{z}_k}}{\mathrm{d}Q_{\Theta_k}}(\boldsymbol{\theta}_k) = \exp\left(-\frac{1}{\lambda_k}\mathsf{L}_k(\boldsymbol{z}_k, \boldsymbol{\theta}_k) - \mathsf{K}_{k, Q_{\Theta_k}, \boldsymbol{z}_k}\left(-\frac{1}{\lambda_k}\right)\right),$$

where the function $\mathsf{K}_{k, Q_{\Theta_k}, \boldsymbol{z}_k}$ is defined as follows

$$\mathsf{K}_{k, Q_{\Theta_k}, \boldsymbol{z}_k} : \begin{cases} \mathbb{R} \longrightarrow \mathbb{R} \\ t \longmapsto \log\left(\int \exp\left(t\, \mathsf{L}_k(\boldsymbol{z}_k, \boldsymbol{\theta}_k)\right) \mathrm{d}Q_{\Theta_k}(\boldsymbol{\theta}_k)\right). \end{cases}$$

Optimization problems

Let $\lambda_k > 0$,

$$\boxed{\min_{P \in \triangle_{Q_{\boldsymbol{\Theta}_k}}(\mathcal{M}_k)} \mathsf{R}_{k,\boldsymbol{z}_k}(P) + \lambda_k \, \mathsf{D}\left(P \| Q_{\boldsymbol{\Theta}_k}\right).}$$

Let $\lambda_k < 0$,

$$\boxed{\max_{P \in \triangle_{Q_{\boldsymbol{\Theta}_k}}(\mathcal{M}_k)} \mathsf{R}_{k,\boldsymbol{z}_k}(P) + \lambda_k \, \mathsf{D}\left(P \| Q_{\boldsymbol{\Theta}_k}\right).}$$

Optimization problems

Let $\lambda_k > 0$,

$$\boxed{\min_{P \in \triangle_{Q_{\boldsymbol{\Theta}_k}}(\mathcal{M}_k)} \mathrm{R}_{k,\boldsymbol{z}_k}(P) + \lambda_k \,\mathrm{D}\left(P \| Q_{\boldsymbol{\Theta}_k}\right).}$$

Let $\gamma \; = \; \mathrm{D}\left(P^{(Q_{\boldsymbol{\Theta}_k}, \lambda_k)}_{\boldsymbol{\Theta}_k | \boldsymbol{Z}_k = \boldsymbol{z}_k} \| Q_{\boldsymbol{\Theta}_k}\right)$,

$$\boxed{\begin{aligned} \min_{P \in \triangle_{Q_{\boldsymbol{\Theta}_k}}(\mathcal{M}_k)} & \mathrm{R}_{k,\boldsymbol{z}_k}(P) \\ \text{s.t. } & \mathrm{D}\left(P \| Q_{\boldsymbol{\Theta}_k}\right) \leq \gamma. \end{aligned}}$$

Let $\lambda_k < 0$,

$$\boxed{\max_{P \in \triangle_{Q_{\boldsymbol{\Theta}_k}}(\mathcal{M}_k)} \mathrm{R}_{k,\boldsymbol{z}_k}(P) + \lambda_k \,\mathrm{D}\left(P \| Q_{\boldsymbol{\Theta}_k}\right).}$$

Let $\gamma \; = \; \mathrm{D}\left(P^{(Q_{\boldsymbol{\Theta}_k}, \lambda_k)}_{\boldsymbol{\Theta}_k | \boldsymbol{Z}_k = \boldsymbol{z}_k} \| Q_{\boldsymbol{\Theta}_k}\right)$,

$$\boxed{\begin{aligned} \max_{P \in \triangle_{Q_{\boldsymbol{\Theta}_k}}(\mathcal{M}_k)} & \mathrm{R}_{k,\boldsymbol{z}_k}(P) \\ \text{s.t. } & \mathrm{D}\left(P \| Q_{\boldsymbol{\Theta}_k}\right) \leq \gamma. \end{aligned}}$$

[5] Samir M. Perlaza and Gaetan Bisson. "**Variations on the expectation due to changes in the probability measure**". Entropy, 27(8:865):1–20, Aug. 2025.

Optimization problems

Let $\lambda_k > 0$,

$$\boxed{\min_{P \in \triangle_{Q_{\boldsymbol{\Theta}_k}}(\mathcal{M}_k)} \mathsf{R}_{k,\boldsymbol{z}_k}(P) + \lambda_k \,\mathsf{D}\left(P \| Q_{\boldsymbol{\Theta}_k}\right).}$$

Let $\gamma \;=\; \mathsf{D}\left(P^{(Q_{\boldsymbol{\Theta}_k},\lambda_k)}_{\boldsymbol{\Theta}_k|\boldsymbol{Z}_k=\boldsymbol{z}_k} \| Q_{\boldsymbol{\Theta}_k}\right)$,

$$\boxed{\begin{array}{rl}\displaystyle\min_{P \in \triangle_{Q_{\boldsymbol{\Theta}_k}}(\mathcal{M}_k)} & \mathsf{R}_{k,\boldsymbol{z}_k}(P) \\ \text{s.t.} & \mathsf{D}\left(P \| Q_{\boldsymbol{\Theta}_k}\right) \leq \gamma.\end{array}}$$

The Best !

Let $\lambda_k < 0$,

$$\boxed{\max_{P \in \triangle_{Q_{\boldsymbol{\Theta}_k}}(\mathcal{M}_k)} \mathsf{R}_{k,\boldsymbol{z}_k}(P) + \lambda_k \,\mathsf{D}\left(P \| Q_{\boldsymbol{\Theta}_k}\right).}$$

Let $\gamma \;=\; \mathsf{D}\left(P^{(Q_{\boldsymbol{\Theta}_k},\lambda_k)}_{\boldsymbol{\Theta}_k|\boldsymbol{Z}_k=\boldsymbol{z}_k} \| Q_{\boldsymbol{\Theta}_k}\right)$,

$$\boxed{\begin{array}{rl}\displaystyle\max_{P \in \triangle_{Q_{\boldsymbol{\Theta}_k}}(\mathcal{M}_k)} & \mathsf{R}_{k,\boldsymbol{z}_k}(P) \\ \text{s.t.} & \mathsf{D}\left(P \| Q_{\boldsymbol{\Theta}_k}\right) \leq \gamma.\end{array}}$$

The Worse !

[5] Samir M. Perlaza and Gaetan Bisson. "**Variations on the expectation due to changes in the probability measure**". Entropy, 27(8:865):1–20, Aug. 2025.

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Generalization Error: The First Closed Form Expression

Theorem

Consider the generalization error $\mathbf{G}_k\left(P_{\underline{\boldsymbol{\Theta}},\underline{\boldsymbol{U}}^{(m)},\underline{\boldsymbol{V}}^{(m)}|\boldsymbol{Z}_0}; P_{\boldsymbol{Z}_0}\right)$ *and assume that for all* $\boldsymbol{z}_0 \in \mathcal{Z}_0$, $P^{(m)}_{\boldsymbol{\Theta}_k} \ll Q_{\boldsymbol{\Theta}_k} \ll P^{(m)}_{\boldsymbol{\Theta}_k} \ll P^{(m)}_{\boldsymbol{\Theta}_k|\boldsymbol{Z}_0=\boldsymbol{z}_0} \ll Q_{\boldsymbol{\Theta}_k}$. *Then,*

$$\mathbf{G}_k\left(P_{\underline{\boldsymbol{\Theta}},\underline{\boldsymbol{U}}^{(m)},\underline{\boldsymbol{V}}^{(m)}|\boldsymbol{Z}_0}; P_{\boldsymbol{Z}_0}\right) = \lambda_k\left(I\left(P^{(m)}_{\boldsymbol{\Theta}_k|\boldsymbol{Z}_0}; P_{\boldsymbol{Z}_0}\right) + L\left(P^{(m)}_{\boldsymbol{\Theta}_k|\boldsymbol{Z}_0}; P_{\boldsymbol{Z}_0}\right) + J_{\mathsf{L}_k,Q_{\boldsymbol{\Theta}_k},\lambda_k}\left(P^{(m)}_{\boldsymbol{\Theta}_k|\boldsymbol{Z}_0}; P_{\boldsymbol{Z}_0}\right)\right).$$

Generalization Error: The First Closed Form Expression

Theorem

Consider the generalization error $\mathbf{G}_k\left(P_{\underline{\boldsymbol{\Theta}}, \underline{\boldsymbol{U}}^{(m)}, \underline{\boldsymbol{V}}^{(m)}|\boldsymbol{Z}_0}; P_{\boldsymbol{Z}_0}\right)$ *and assume that for all* $\boldsymbol{z}_0 \in \mathcal{Z}_0$, $P^{(m)}_{\boldsymbol{\Theta}_k} \ll Q_{\boldsymbol{\Theta}_k} \ll P^{(m)}_{\boldsymbol{\Theta}_k} \ll P^{(m)}_{\boldsymbol{\Theta}_k|\boldsymbol{Z}_0=\boldsymbol{z}_0} \ll Q_{\boldsymbol{\Theta}_k}$. *Then,*

$$\mathbf{G}_k\left(P_{\underline{\boldsymbol{\Theta}}, \underline{\boldsymbol{U}}^{(m)}, \underline{\boldsymbol{V}}^{(m)}|\boldsymbol{Z}_0}; P_{\boldsymbol{Z}_0}\right) = \lambda_k\left(I\left(P^{(m)}_{\boldsymbol{\Theta}_k|\boldsymbol{Z}_0}; P_{\boldsymbol{Z}_0}\right) + L\left(P^{(m)}_{\boldsymbol{\Theta}_k|\boldsymbol{Z}_0}; P_{\boldsymbol{Z}_0}\right) + J_{\mathsf{L}_k, Q_{\boldsymbol{\Theta}_k}, \lambda_k}\left(P^{(m)}_{\boldsymbol{\Theta}_k|\boldsymbol{Z}_0}; P_{\boldsymbol{Z}_0}\right)\right).$$

$$J_{\mathsf{L}_k, Q_{\boldsymbol{\Theta}_k}, \lambda_k}\left(P^{(m)}_{\boldsymbol{\Theta}_k|\boldsymbol{Z}_0}; P_{\boldsymbol{Z}_0}\right)$$
$$= \int \log\left(\frac{\mathrm{d}P^{(m)}_{\boldsymbol{\Theta}_k|\boldsymbol{Z}_0=\boldsymbol{z}_0}}{\mathrm{d}P^{(Q_{\boldsymbol{\Theta}_k}, \lambda_k)}_{\boldsymbol{\Theta}_k|\boldsymbol{Z}_k=\boldsymbol{z}_k}}(\boldsymbol{\theta}_k)\right)\mathrm{d}P^{(m)}_{\boldsymbol{\Theta}_k}P_{\boldsymbol{Z}_0}(\boldsymbol{\theta}_k, \boldsymbol{z}_0) - \int \log\left(\frac{\mathrm{d}P^{(m)}_{\boldsymbol{\Theta}_k|\boldsymbol{Z}_0=\boldsymbol{z}_0}}{\mathrm{d}P^{(Q_{\boldsymbol{\Theta}_k}, \lambda_k)}_{\boldsymbol{\Theta}_k|\boldsymbol{Z}_k=\boldsymbol{z}_k}}(\boldsymbol{\theta}_k)\right)\mathrm{d}P^{(m)}_{\boldsymbol{\Theta}_k|\boldsymbol{Z}_0}P_{\boldsymbol{Z}_0}(\boldsymbol{\theta}_k, \boldsymbol{z}_0),$$

where the probability measure $P^{(m)}_{\boldsymbol{\Theta}_k}$ is such that for all measurable sets $\mathcal{A} \subseteq \mathcal{M}_k$

$$P^{(m)}_{\boldsymbol{\Theta}_k}(\mathcal{A}) = \int P^{(m)}_{\boldsymbol{\Theta}_k|\boldsymbol{Z}_0=\boldsymbol{z}_0}(\mathcal{A})\,\mathrm{d}P_{\boldsymbol{Z}_0}(\boldsymbol{z}_0);$$

$P^{(m)}_{\boldsymbol{\Theta}_k}P_{\boldsymbol{Z}_0} \in \triangle(\mathcal{M}_k \times \mathcal{Z}_0)$ is a product measure formed by $P^{(m)}_{\boldsymbol{\Theta}_k}$ and $P_{\boldsymbol{Z}_0}$; and
$P^{(m)}_{\boldsymbol{\Theta}_k|\boldsymbol{Z}_0}P_{\boldsymbol{Z}_0} \in \triangle(\mathcal{M}_k \times \}setZ_0)$ is the joint probability measure induced by $P^{(m)}_{\boldsymbol{\Theta}_k|\boldsymbol{Z}_0}$.

Generalization Error: The First Closed Form Expression

Theorem

Consider the generalization error $\mathbf{G}_k\left(P_{\underline{\boldsymbol{\Theta}}, \underline{\boldsymbol{U}}^{(m)}, \underline{\boldsymbol{V}}^{(m)}|\boldsymbol{Z}_0}; P_{\boldsymbol{Z}_0}\right)$ *and assume that for all* $\boldsymbol{z}_0 \in \mathcal{Z}_0$, $P^{(m)}_{\boldsymbol{\Theta}_k} \ll Q_{\boldsymbol{\Theta}_k} \ll P^{(m)}_{\boldsymbol{\Theta}_k} \ll P^{(m)}_{\boldsymbol{\Theta}_k|\boldsymbol{Z}_0=\boldsymbol{z}_0} \ll Q_{\boldsymbol{\Theta}_k}$. *Then,*

$$\mathbf{G}_k\left(P_{\underline{\boldsymbol{\Theta}}, \underline{\boldsymbol{U}}^{(m)}, \underline{\boldsymbol{V}}^{(m)}|\boldsymbol{Z}_0}; P_{\boldsymbol{Z}_0}\right) = \lambda_k\left(I\left(P^{(m)}_{\boldsymbol{\Theta}_k|\boldsymbol{Z}_0}; P_{\boldsymbol{Z}_0}\right) + L\left(P^{(m)}_{\boldsymbol{\Theta}_k|\boldsymbol{Z}_0}; P_{\boldsymbol{Z}_0}\right) + J_{\mathsf{L}_k, Q_{\boldsymbol{\Theta}_k}, \lambda_k}\left(P^{(m)}_{\boldsymbol{\Theta}_k|\boldsymbol{Z}_0}; P_{\boldsymbol{Z}_0}\right)\right).$$

Warnings :

- λ_k does not influence the value of $\mathbf{G}_k\left(P_{\underline{\boldsymbol{\Theta}}, \underline{\boldsymbol{U}}^{(m)}, \underline{\boldsymbol{V}}^{(m)}|\boldsymbol{Z}_0}; P_{\boldsymbol{Z}_0}\right)$;
- $I\left(P^{(m)}_{\boldsymbol{\Theta}_k|\boldsymbol{Z}_0}; P_{\boldsymbol{Z}_0}\right) + L\left(P^{(m)}_{\boldsymbol{\Theta}_k|\boldsymbol{Z}_0}; P_{\boldsymbol{Z}_0}\right) \geq 0$;
- $J_{\mathsf{L}_k, Q_{\boldsymbol{\Theta}_k}, \lambda_k}\left(P^{(m)}_{\boldsymbol{\Theta}_k|\boldsymbol{Z}_0}; P_{\boldsymbol{Z}_0}\right) \in \mathbb{R}$.

Independence Hypothesis Test

$$
\begin{aligned}
& I\left(P^{(m)}_{\boldsymbol{\Theta}_k|\boldsymbol{Z}_0}; P_{\boldsymbol{Z}_0}\right) + L\left(P^{(m)}_{\boldsymbol{\Theta}_k|\boldsymbol{Z}_0}; P_{\boldsymbol{Z}_0}\right) \\
&= \int \log\left(\frac{\mathrm{d}P^{(m)}_{\boldsymbol{\Theta}_k|\boldsymbol{Z}_0=\boldsymbol{z}_0}}{\mathrm{d}P^{(m)}_{\boldsymbol{\Theta}_k}}(\boldsymbol{\theta}_k)\right)\mathrm{d}P^{(m)}_{\boldsymbol{\Theta}_k|\boldsymbol{Z}_0}P_{\boldsymbol{Z}_0}(\boldsymbol{\theta}_k, \boldsymbol{z}_0) - \int \log\left(\frac{\mathrm{d}P^{(m)}_{\boldsymbol{\Theta}_k|\boldsymbol{Z}_0=\boldsymbol{z}_0}}{\mathrm{d}P^{(m)}_{\boldsymbol{\Theta}_k}}(\boldsymbol{\theta}_k)\right)\mathrm{d}P^{(m)}_{\boldsymbol{\Theta}_k}P_{\boldsymbol{Z}_0}(\boldsymbol{\theta}_k, \boldsymbol{z}_0)
\end{aligned}
$$

Independence Hypothesis Test

$$
\begin{aligned}
&I\left(P^{(m)}_{\boldsymbol{\Theta}_k|\boldsymbol{Z}_0}; P_{\boldsymbol{Z}_0}\right) + L\left(P^{(m)}_{\boldsymbol{\Theta}_k|\boldsymbol{Z}_0}; P_{\boldsymbol{Z}_0}\right) \\
&= \int \log\left(\frac{\mathrm{d}P^{(m)}_{\boldsymbol{\Theta}_k|\boldsymbol{Z}_0=\boldsymbol{z}_0}}{\mathrm{d}P^{(m)}_{\boldsymbol{\Theta}_k}}(\boldsymbol{\theta}_k)\right)\mathrm{d}P^{(m)}_{\boldsymbol{\Theta}_k|\boldsymbol{Z}_0}P_{\boldsymbol{Z}_0}(\boldsymbol{\theta}_k,\boldsymbol{z}_0) - \int \log\left(\frac{\mathrm{d}P^{(m)}_{\boldsymbol{\Theta}_k|\boldsymbol{Z}_0=\boldsymbol{z}_0}}{\mathrm{d}P^{(m)}_{\boldsymbol{\Theta}_k}}(\boldsymbol{\theta}_k)\right)\mathrm{d}P^{(m)}_{\boldsymbol{\Theta}_k}P_{\boldsymbol{Z}_0}(\boldsymbol{\theta}_k,\boldsymbol{z}_0)
\end{aligned}
$$

Independence Hypothesis Test

$$I\left(P^{(m)}_{\boldsymbol{\Theta}_k|\boldsymbol{Z}_0}; P_{\boldsymbol{Z}_0}\right) + L\left(P^{(m)}_{\boldsymbol{\Theta}_k|\boldsymbol{Z}_0}; P_{\boldsymbol{Z}_0}\right)$$
$$= \int \log\left(\frac{\mathrm{d}P^{(m)}_{\boldsymbol{\Theta}_k|\boldsymbol{Z}_0=\boldsymbol{z}_0}}{\mathrm{d}P^{(m)}_{\boldsymbol{\Theta}_k}}(\boldsymbol{\theta}_k)\right)\mathrm{d}P^{(m)}_{\boldsymbol{\Theta}_k|\boldsymbol{Z}_0}P_{\boldsymbol{Z}_0}(\boldsymbol{\theta}_k,\boldsymbol{z}_0) - \int \log\left(\frac{\mathrm{d}P^{(m)}_{\boldsymbol{\Theta}_k|\boldsymbol{Z}_0=\boldsymbol{z}_0}}{\mathrm{d}P^{(m)}_{\boldsymbol{\Theta}_k}}(\boldsymbol{\theta}_k)\right)\mathrm{d}P^{(m)}_{\boldsymbol{\Theta}_k}P_{\boldsymbol{Z}_0}(\boldsymbol{\theta}_k,\boldsymbol{z}_0)$$

Hypothesis:

$$H_0 : (\boldsymbol{\theta}_k, \boldsymbol{z}_0) \sim P^{(m)}_{\boldsymbol{\Theta}_k|\boldsymbol{Z}_0}P_{\boldsymbol{Z}_0}$$

$$H_1 : (\boldsymbol{\theta}_k, \boldsymbol{z}_0) \sim P^{(m)}_{\boldsymbol{\Theta}_k}P_{\boldsymbol{Z}_0}$$

Contribution of the Federation

$$
\begin{aligned}
& J_{\mathsf{L}_k, Q_{\boldsymbol{\Theta}_k}, \lambda_k}\left(P^{(m)}_{\boldsymbol{\Theta}_k|\boldsymbol{Z}_0}; P_{\boldsymbol{Z}_0}\right) \\
= & \int \log\left(\frac{\mathrm{d}P^{(m)}_{\boldsymbol{\Theta}_k|\boldsymbol{Z}_0=\boldsymbol{z}_0}}{\mathrm{d}P^{(Q_{\boldsymbol{\Theta}_k},\lambda_k)}_{\boldsymbol{\Theta}_k|\boldsymbol{Z}_k=\boldsymbol{z}_k}}(\boldsymbol{\theta}_k)\right)\mathrm{d}P^{(m)}_{\boldsymbol{\Theta}_k}P_{\boldsymbol{Z}_0}(\boldsymbol{\theta}_k,\boldsymbol{z}_0) - \int \log\left(\frac{\mathrm{d}P^{(m)}_{\boldsymbol{\Theta}_k|\boldsymbol{Z}_0=\boldsymbol{z}_0}}{\mathrm{d}P^{(Q_{\boldsymbol{\Theta}_k},\lambda_k)}_{\boldsymbol{\Theta}_k|\boldsymbol{Z}_k=\boldsymbol{z}_k}}(\boldsymbol{\theta}_k)\right)\mathrm{d}P^{(m)}_{\boldsymbol{\Theta}_k|\boldsymbol{Z}_0}P_{\boldsymbol{Z}_0}(\boldsymbol{\theta}_k,\boldsymbol{z}_0),
\end{aligned}
$$

Contribution of the Federation

$$
\begin{aligned}
& J_{\mathsf{L}_k, Q_{\boldsymbol{\Theta}_k}, \lambda_k}\left(P^{(m)}_{\boldsymbol{\Theta}_k|\boldsymbol{Z}_0}; P_{\boldsymbol{Z}_0}\right) \\
&= \int \log\left(\frac{\mathrm{d}P^{(m)}_{\boldsymbol{\Theta}_k|\boldsymbol{Z}_0=\boldsymbol{z}_0}}{\mathrm{d}P^{(Q_{\boldsymbol{\Theta}_k},\lambda_k)}_{\boldsymbol{\Theta}_k|\boldsymbol{Z}_k=\boldsymbol{z}_k}}(\boldsymbol{\theta}_k)\right)\mathrm{d}P^{(m)}_{\boldsymbol{\Theta}_k}P_{\boldsymbol{Z}_0}(\boldsymbol{\theta}_k,\boldsymbol{z}_0) - \int \log\left(\frac{\mathrm{d}P^{(m)}_{\boldsymbol{\Theta}_k|\boldsymbol{Z}_0=\boldsymbol{z}_0}}{\mathrm{d}P^{(Q_{\boldsymbol{\Theta}_k},\lambda_k)}_{\boldsymbol{\Theta}_k|\boldsymbol{Z}_k=\boldsymbol{z}_k}}(\boldsymbol{\theta}_k)\right)\mathrm{d}P^{(m)}_{\boldsymbol{\Theta}_k|\boldsymbol{Z}_0}P_{\boldsymbol{Z}_0}(\boldsymbol{\theta}_k,\boldsymbol{z}_0),
\end{aligned}
$$

Mismatch Hypothesis testing:

$$H_0 : (\boldsymbol{\theta}_k, \boldsymbol{z}_0) \sim P^{(m)}_{\boldsymbol{\Theta}_k|\boldsymbol{Z}_0} P_{\boldsymbol{Z}_0}$$

$$H_1 : (\boldsymbol{\theta}_k, \boldsymbol{z}_0) \sim P^{(Q_{\boldsymbol{\Theta}_k},\lambda_k)}_{\boldsymbol{\Theta}_k|\boldsymbol{Z}_k} P_{\boldsymbol{Z}_0}$$

Contribution of the Federation

$$
\begin{aligned}
&J_{\mathsf{L}_k, Q_{\boldsymbol{\Theta}_k}, \lambda_k}\left(P^{(m)}_{\boldsymbol{\Theta}_k|\boldsymbol{Z}_0}; P_{\boldsymbol{Z}_0}\right) \\
&= \int \log\left(\frac{\mathrm{d}P^{(m)}_{\boldsymbol{\Theta}_k|\boldsymbol{Z}_0=\boldsymbol{z}_0}}{\mathrm{d}P^{(Q_{\boldsymbol{\Theta}_k},\lambda_k)}_{\boldsymbol{\Theta}_k|\boldsymbol{Z}_k=\boldsymbol{z}_k}}(\boldsymbol{\theta}_k)\right)\mathrm{d}P^{(m)}_{\boldsymbol{\Theta}_k}P_{\boldsymbol{Z}_0}(\boldsymbol{\theta}_k,\boldsymbol{z}_0) - \int \log\left(\frac{\mathrm{d}P^{(m)}_{\boldsymbol{\Theta}_k|\boldsymbol{Z}_0=\boldsymbol{z}_0}}{\mathrm{d}P^{(Q_{\boldsymbol{\Theta}_k},\lambda_k)}_{\boldsymbol{\Theta}_k|\boldsymbol{Z}_k=\boldsymbol{z}_k}}(\boldsymbol{\theta}_k)\right)\mathrm{d}P^{(m)}_{\boldsymbol{\Theta}_k|\boldsymbol{Z}_0}P_{\boldsymbol{Z}_0}(\boldsymbol{\theta}_k,\boldsymbol{z}_0),
\end{aligned}
$$

Mismatch Hypothesis testing:

$$H_0 : (\boldsymbol{\theta}_k, \boldsymbol{z}_0) \sim P^{(m)}_{\boldsymbol{\Theta}_k|\boldsymbol{Z}_0}P_{\boldsymbol{Z}_0}$$

$$H_1 : (\boldsymbol{\theta}_k, \boldsymbol{z}_0) \sim P^{(Q_{\boldsymbol{\Theta}_k},\lambda_k)}_{\boldsymbol{\Theta}_k|\boldsymbol{Z}_k}P_{\boldsymbol{Z}_0}$$

Contribution of the Federation

$$J_{\mathsf{L}_k, Q_{\boldsymbol{\Theta}_k}, \lambda_k}\left(P^{(m)}_{\boldsymbol{\Theta}_k|\boldsymbol{Z}_0}; P_{\boldsymbol{Z}_0}\right)$$
$$= \int \log\left(\frac{\mathrm{d}P^{(m)}_{\boldsymbol{\Theta}_k|\boldsymbol{Z}_0=\boldsymbol{z}_0}}{\mathrm{d}P^{(Q_{\boldsymbol{\Theta}_k},\lambda_k)}_{\boldsymbol{\Theta}_k|\boldsymbol{Z}_k=\boldsymbol{z}_k}}(\boldsymbol{\theta}_k)\right)\mathrm{d}P^{(m)}_{\boldsymbol{\Theta}_k} P_{\boldsymbol{Z}_0}(\boldsymbol{\theta}_k,\boldsymbol{z}_0) - \int \log\left(\frac{\mathrm{d}P^{(m)}_{\boldsymbol{\Theta}_k|\boldsymbol{Z}_0=\boldsymbol{z}_0}}{\mathrm{d}P^{(Q_{\boldsymbol{\Theta}_k},\lambda_k)}_{\boldsymbol{\Theta}_k|\boldsymbol{Z}_k=\boldsymbol{z}_k}}(\boldsymbol{\theta}_k)\right)\mathrm{d}P^{(m)}_{\boldsymbol{\Theta}_k|\boldsymbol{Z}_0} P_{\boldsymbol{Z}_0}(\boldsymbol{\theta}_k,\boldsymbol{z}_0),$$

Mismatch Hypothesis testing:

$$H_0 : (\boldsymbol{\theta}_k, \boldsymbol{z}_0) \sim P^{(m)}_{\boldsymbol{\Theta}_k|\boldsymbol{Z}_0} P_{\boldsymbol{Z}_0}$$

$$H_1 : (\boldsymbol{\theta}_k, \boldsymbol{z}_0) \sim P^{(Q_{\boldsymbol{\Theta}_k},\lambda_k)}_{\boldsymbol{\Theta}_k|\boldsymbol{Z}_k} P_{\boldsymbol{Z}_0}$$

Contribution of the Federation

$$
\begin{aligned}
&J_{\mathsf{L}_k, Q_{\boldsymbol{\Theta}_k}, \lambda_k}\left(P^{(m)}_{\boldsymbol{\Theta}_k|\boldsymbol{Z}_0}; P_{\boldsymbol{Z}_0}\right) \\
&= \int \log\left(\frac{\mathrm{d}P^{(m)}_{\boldsymbol{\Theta}_k|\boldsymbol{Z}_0=\boldsymbol{z}_0}}{\mathrm{d}P^{(Q_{\boldsymbol{\Theta}_k},\lambda_k)}_{\boldsymbol{\Theta}_k|\boldsymbol{Z}_k=\boldsymbol{z}_k}}(\boldsymbol{\theta}_k)\right)\mathrm{d}P^{(m)}_{\boldsymbol{\Theta}_k}P_{\boldsymbol{Z}_0}(\boldsymbol{\theta}_k,\boldsymbol{z}_0) - \int \log\left(\frac{\mathrm{d}P^{(m)}_{\boldsymbol{\Theta}_k|\boldsymbol{Z}_0=\boldsymbol{z}_0}}{\mathrm{d}P^{(Q_{\boldsymbol{\Theta}_k},\lambda_k)}_{\boldsymbol{\Theta}_k|\boldsymbol{Z}_k=\boldsymbol{z}_k}}(\boldsymbol{\theta}_k)\right)\mathrm{d}P^{(m)}_{\boldsymbol{\Theta}_k|\boldsymbol{Z}_0}P_{\boldsymbol{Z}_0}(\boldsymbol{\theta}_k,\boldsymbol{z}_0),
\end{aligned}
$$

Consider $\gamma > 0$,

$$
\mathcal{A}_\gamma = \left\{(\boldsymbol{\theta}_k, \boldsymbol{z}_0) \in \mathcal{M}_k \times \mathcal{Z}_0 : \log\left(\frac{\mathrm{d}P^{(m)}_{\boldsymbol{\Theta}_k|\boldsymbol{Z}_0=\boldsymbol{z}_0}}{\mathrm{d}P^{(Q_{\boldsymbol{\Theta}_k},\lambda_k)}_{\boldsymbol{\Theta}_k|\boldsymbol{Z}_k=\boldsymbol{z}_k}}(\boldsymbol{\theta}_k)\right) \geqslant \gamma\right\}
$$

The set $\mathcal{A}_\gamma$ is the acceptance region of $H_0 : (\boldsymbol{\theta}_k, \boldsymbol{z}_0) \sim P^{(m)}_{\boldsymbol{\Theta}_k|\boldsymbol{Z}_0}P_{\boldsymbol{Z}_0}$.

The set $\mathcal{A}^{\mathsf{c}}_\gamma$ is the acceptance region of $H_1 : (\boldsymbol{\theta}_k, \boldsymbol{z}_0) \sim P^{(Q_{\boldsymbol{\Theta}_k},\lambda_k)}_{\boldsymbol{\Theta}_k|\boldsymbol{Z}_k}P_{\boldsymbol{Z}_0}$.

Consider $\gamma > 0$,

$$\mathcal{A}_\gamma = \left\{ (\boldsymbol{\theta}_k, \boldsymbol{z}_0) \in \mathcal{M}_k \times \mathcal{Z}_0 : \log\left(\frac{\mathrm{d}P^{(m)}_{\boldsymbol{\Theta}_k|\boldsymbol{Z}_0=\boldsymbol{z}_0}}{\mathrm{d}P^{(Q_{\boldsymbol{\Theta}_k},\lambda_k)}_{\boldsymbol{\Theta}_k|\boldsymbol{Z}_k=\boldsymbol{z}_k}} (\boldsymbol{\theta}_k) \right) \geqslant \gamma \right\}$$

The set $\mathcal{A}_\gamma$ is the acceptance region of $H_0 : (\boldsymbol{\theta}_k, \boldsymbol{z}_0) \sim P^{(m)}_{\boldsymbol{\Theta}_k|\boldsymbol{Z}_0} P_{\boldsymbol{Z}_0}$.

The set $\mathcal{A}^{\mathrm{c}}_\gamma$ is the acceptance region of $H_1 : (\boldsymbol{\theta}_k, \boldsymbol{z}_0) \sim P^{(Q_{\boldsymbol{\Theta}_k},\lambda_k)}_{\boldsymbol{\Theta}_k|\boldsymbol{Z}_k} P_{\boldsymbol{Z}_0}$.

Consider $\gamma > 0$,

$$\mathcal{A}_\gamma = \left\{ (\boldsymbol{\theta}_k, \boldsymbol{z}_0) \in \mathcal{M}_k \times \mathcal{Z}_0 : \log \left(\frac{\mathrm{d}P^{(m)}_{\boldsymbol{\Theta}_k | \boldsymbol{Z}_0 = \boldsymbol{z}_0}}{\mathrm{d}P^{(Q_{\boldsymbol{\Theta}_k}, \lambda_k)}_{\boldsymbol{\Theta}_k | \boldsymbol{Z}_k = \boldsymbol{z}_k}} (\boldsymbol{\theta}_k) \right) \geqslant \gamma \right\}$$

The set $\mathcal{A}_\gamma$ is the acceptance region of $H_0 : (\boldsymbol{\theta}_k, \boldsymbol{z}_0) \sim P^{(m)}_{\boldsymbol{\Theta}_k | \boldsymbol{Z}_0} P_{\boldsymbol{Z}_0}$.

The set $\mathcal{A}_\gamma^{\mathrm{c}}$ is the acceptance region of $H_1 : (\boldsymbol{\theta}_k, \boldsymbol{z}_0) \sim P^{(Q_{\boldsymbol{\Theta}_k}, \lambda_k)}_{\boldsymbol{\Theta}_k | \boldsymbol{Z}_k} P_{\boldsymbol{Z}_0}$.

$P^{(m)}_{\boldsymbol{\Theta}_k} P_{\boldsymbol{Z}_0} (\mathcal{A}_\gamma)$

Consider $\gamma > 0$,

$$\mathcal{A}_\gamma = \left\{ (\boldsymbol{\theta}_k, \boldsymbol{z}_0) \in \mathcal{M}_k \times \mathcal{Z}_0 : \log\left(\frac{\mathrm{d}P^{(m)}_{\boldsymbol{\Theta}_k | \boldsymbol{Z}_0 = \boldsymbol{z}_0}}{\mathrm{d}P^{(Q_{\boldsymbol{\Theta}_k}, \lambda_k)}_{\boldsymbol{\Theta}_k | \boldsymbol{Z}_k = \boldsymbol{z}_k}} (\boldsymbol{\theta}_k) \right) \geqslant \gamma \right\}$$

The set $\mathcal{A}_\gamma$ is the acceptance region of $H_0 : (\boldsymbol{\theta}_k, \boldsymbol{z}_0) \sim P^{(m)}_{\boldsymbol{\Theta}_k | \boldsymbol{Z}_0} P_{\boldsymbol{Z}_0}$.

The set $\mathcal{A}_\gamma^{\mathrm{c}}$ is the acceptance region of $H_1 : (\boldsymbol{\theta}_k, \boldsymbol{z}_0) \sim P^{(Q_{\boldsymbol{\Theta}_k}, \lambda_k)}_{\boldsymbol{\Theta}_k | \boldsymbol{Z}_k} P_{\boldsymbol{Z}_0}$.

$$P^{(m)}_{\boldsymbol{\Theta}_k} P_{\boldsymbol{Z}_0} (\mathcal{A}_\gamma)$$

$$P^{(m)}_{\boldsymbol{\Theta}_k | \boldsymbol{Z}_0} P_{\boldsymbol{Z}_0} (\mathcal{A}_\gamma)$$

Consider $\gamma > 0$,

$$\mathcal{A}_\gamma = \left\{ (\boldsymbol{\theta}_k, \boldsymbol{z}_0) \in \mathcal{M}_k \times \mathcal{Z}_0 : \log\left(\frac{\mathrm{d}P^{(m)}_{\boldsymbol{\Theta}_k | \boldsymbol{Z}_0 = \boldsymbol{z}_0}}{\mathrm{d}P^{(Q_{\boldsymbol{\Theta}_k}, \lambda_k)}_{\boldsymbol{\Theta}_k | \boldsymbol{Z}_k = \boldsymbol{z}_k}} (\boldsymbol{\theta}_k) \right) \geqslant \gamma \right\}$$

The set $\mathcal{A}_\gamma$ is the acceptance region of $H_0 : (\boldsymbol{\theta}_k, \boldsymbol{z}_0) \sim P^{(m)}_{\boldsymbol{\Theta}_k | \boldsymbol{Z}_0} P_{\boldsymbol{Z}_0}$.

The set $\mathcal{A}^{\mathrm{c}}_\gamma$ is the acceptance region of $H_1 : (\boldsymbol{\theta}_k, \boldsymbol{z}_0) \sim P^{(Q_{\boldsymbol{\Theta}_k}, \lambda_k)}_{\boldsymbol{\Theta}_k | \boldsymbol{Z}_k} P_{\boldsymbol{Z}_0}$.

$$P^{(m)}_{\boldsymbol{\Theta}_k} P_{\boldsymbol{Z}_0} (\mathcal{A}_\gamma)$$

$$P^{(m)}_{\boldsymbol{\Theta}_k | \boldsymbol{Z}_0} P_{\boldsymbol{Z}_0} (\mathcal{A}_\gamma)$$

$$P^{(m)}_{\boldsymbol{\Theta}_k} P_{\boldsymbol{Z}_0} (\mathcal{A}^{\mathrm{c}}_\gamma)$$

Consider $\gamma > 0$,

$$\mathcal{A}_\gamma = \left\{ (\boldsymbol{\theta}_k, \boldsymbol{z}_0) \in \mathcal{M}_k \times \mathcal{Z}_0 : \log \left(\frac{\mathrm{d}P^{(m)}_{\boldsymbol{\Theta}_k | \boldsymbol{Z}_0 = \boldsymbol{z}_0}}{\mathrm{d}P^{(Q_{\boldsymbol{\Theta}_k}, \lambda_k)}_{\boldsymbol{\Theta}_k | \boldsymbol{Z}_k = \boldsymbol{z}_k}} (\boldsymbol{\theta}_k) \right) \geqslant \gamma \right\}$$

The set $\mathcal{A}_\gamma$ is the acceptance region of $H_0 : (\boldsymbol{\theta}_k, \boldsymbol{z}_0) \sim P^{(m)}_{\boldsymbol{\Theta}_k | \boldsymbol{Z}_0} P_{\boldsymbol{Z}_0}$.

The set $\mathcal{A}^{\mathrm{c}}_\gamma$ is the acceptance region of $H_1 : (\boldsymbol{\theta}_k, \boldsymbol{z}_0) \sim P^{(Q_{\boldsymbol{\Theta}_k}, \lambda_k)}_{\boldsymbol{\Theta}_k | \boldsymbol{Z}_k} P_{\boldsymbol{Z}_0}$.

$P^{(m)}_{\boldsymbol{\Theta}_k} P_{\boldsymbol{Z}_0} (\mathcal{A}_\gamma)$	$P^{(m)}_{\boldsymbol{\Theta}_k} P_{\boldsymbol{Z}_0} (\mathcal{A}^{\mathrm{c}}_\gamma)$
$P^{(m)}_{\boldsymbol{\Theta}_k \| \boldsymbol{Z}_0} P_{\boldsymbol{Z}_0} (\mathcal{A}_\gamma)$	$P^{(m)}_{\boldsymbol{\Theta}_k \| \boldsymbol{Z}_0} P_{\boldsymbol{Z}_0} (\mathcal{A}^{\mathrm{c}}_\gamma)$

Bounds in Probability

Theorem

Assume that for all $\boldsymbol{z}_0 \in \mathcal{Z}_0$, $P^{(m)}_{\boldsymbol{\Theta}_k|\boldsymbol{Z}_0=\boldsymbol{z}_0} \ll Q_{\boldsymbol{\Theta}_k}$*; consider a free parameter* $\gamma > 0$*; the following constants*

$$\underline{\gamma} \triangleq \min_{(\boldsymbol{\theta}_k,\boldsymbol{z}_0)\in\mathcal{M}_k\times\mathcal{Z}_0} \log\left(\frac{\mathrm{d}P^{(m)}_{\boldsymbol{\Theta}_k|\boldsymbol{Z}_0=\boldsymbol{z}_0}}{\mathrm{d}P^{(Q_{\boldsymbol{\Theta}_k},\lambda_k)}_{\boldsymbol{\Theta}_k|\boldsymbol{Z}_k=\boldsymbol{z}_k}}(\boldsymbol{\theta}_k)\right) \text{ and } \bar{\gamma} \triangleq \max_{(\boldsymbol{\theta}_k,\boldsymbol{z}_0)\in\mathcal{M}_k\times\mathcal{Z}_0} \log\left(\frac{\mathrm{d}P^{(m)}_{\boldsymbol{\Theta}_k|\boldsymbol{Z}_0=\boldsymbol{z}_0}}{\mathrm{d}P^{(Q_{\boldsymbol{\Theta}_k},\lambda_k)}_{\boldsymbol{\Theta}_k|\boldsymbol{Z}_k=\boldsymbol{z}_k}}(\boldsymbol{\theta}_k)\right);$$

and the set $\mathcal{A}_\gamma = \left\{(\boldsymbol{\theta}_k,\boldsymbol{z}_0)\in\mathcal{M}_k\times\mathcal{Z}_0 : \log\left(\frac{\mathrm{d}P^{(m)}_{\boldsymbol{\Theta}_k|\boldsymbol{Z}_0=\boldsymbol{z}_0}}{\mathrm{d}P^{(Q_{\boldsymbol{\Theta}_k},\lambda_k)}_{\boldsymbol{\Theta}_k|\boldsymbol{Z}_k=\boldsymbol{z}_k}}(\boldsymbol{\theta}_k)\right) \geqslant \gamma\right\}$. *Then, the value* $J_{\mathsf{L}_k,Q_{\boldsymbol{\Theta}_k},\lambda_k}\left(P^{(m)}_{\boldsymbol{\Theta}_k|\boldsymbol{Z}_0}; P_{\boldsymbol{Z}_0}\right)$*, satisfies*

$$\begin{aligned}
&P^{(m)}_{\boldsymbol{\Theta}_k}P_{\boldsymbol{Z}_0}(\mathcal{A}_\gamma)(\bar{\gamma}-\gamma) + P^{(m)}_{\boldsymbol{\Theta}_k|\boldsymbol{Z}_0}P_{\boldsymbol{Z}_0}(\mathcal{A}^c_\gamma)\left(\gamma-\underline{\gamma}\right)\\
&\geqslant J_{\mathsf{L}_k,Q_{\boldsymbol{\Theta}_k},\lambda_k}\left(P^{(m)}_{\boldsymbol{\Theta}_k|\boldsymbol{Z}_0}; P_{\boldsymbol{Z}_0}\right) \geqslant\\
&P^{(m)}_{\boldsymbol{\Theta}_k}P_{\boldsymbol{Z}_0}(\mathcal{A}^c_\gamma)\left(\underline{\gamma}-\gamma\right) + P^{(m)}_{\boldsymbol{\Theta}_k|\boldsymbol{Z}_0}P_{\boldsymbol{Z}_0}(\mathcal{A}_\gamma)(\gamma-\bar{\gamma}).
\end{aligned}$$

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Generalization Error: The Second Closed Form Expression

Theorem

Consider the generalization error $\mathbf{G}_k\left(P_{\underline{\boldsymbol{\Theta}}, \underline{\boldsymbol{U}}^{(m)}, \underline{\boldsymbol{V}}^{(m)} | \boldsymbol{Z}_0}; P_{\boldsymbol{Z}_0}\right)$ *and assume that for all* $\boldsymbol{z}_0 \in \mathcal{Z}_0$, $P^{(m)}_{\boldsymbol{\Theta}_k | \boldsymbol{Z}_0 = \boldsymbol{z}_0} \ll Q_{\boldsymbol{\Theta}_k}$. *Then,*

$$\begin{aligned} &\mathbf{G}_k\left(P_{\underline{\boldsymbol{\Theta}}, \underline{\boldsymbol{U}}^{(m)}, \underline{\boldsymbol{V}}^{(m)} | \boldsymbol{Z}_0}; P_{\boldsymbol{Z}_0}\right) \\ &= \lambda_k \iint \left(\mathrm{D}\left(P^{(m)}_{\boldsymbol{\Theta}_k | \boldsymbol{Z}_0 = \boldsymbol{z}_0} \| P^{(Q_{\boldsymbol{\Theta}_k}, \lambda_k)}_{\boldsymbol{\Theta}_k | \boldsymbol{Z}_k = \hat{\boldsymbol{z}}_k}\right) - \mathrm{D}\left(P^{(m)}_{\boldsymbol{\Theta}_k | \boldsymbol{Z}_0 = \boldsymbol{z}_0} \| P^{(Q_{\boldsymbol{\Theta}_k}, \lambda_k)}_{\boldsymbol{\Theta}_k | \boldsymbol{Z}_k = \boldsymbol{z}_k}\right)\right) \mathrm{d}P_{\boldsymbol{Z}_k}(\hat{\boldsymbol{z}}_k) \, \mathrm{d}P_{\boldsymbol{Z}_0}(\boldsymbol{z}_0). \end{aligned}$$

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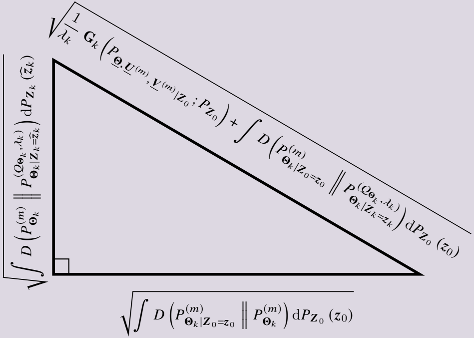
Generalization Error: A Pythagorean Identity

Theorem

Consider the generalization error $\mathbf{G}_k\left(P_{\underline{\boldsymbol{\Theta}},\underline{\boldsymbol{U}}^{(m)},\underline{\boldsymbol{V}}^{(m)}|\boldsymbol{Z}_0};P_{\boldsymbol{Z}_0}\right)$. *Assume that for all* $\boldsymbol{z}_0 \in \mathcal{Z}_0$, $P^{(m)}_{\boldsymbol{\Theta}_k|\boldsymbol{Z}_0=\boldsymbol{z}_0} \ll P^{(m)}_{\boldsymbol{\Theta}_k} \ll Q_{\boldsymbol{\Theta}_k}$. *Then,*

$$
\begin{aligned}
&\frac{1}{\lambda_k}\mathbf{G}_k\left(P_{\underline{\boldsymbol{\Theta}},\underline{\boldsymbol{U}}^{(m)},\underline{\boldsymbol{V}}^{(m)}|\boldsymbol{Z}_0};P_{\boldsymbol{Z}_0}\right)\\
&\quad+\int \mathrm{D}\left(P^{(m)}_{\boldsymbol{\Theta}_k|\boldsymbol{Z}_0=\boldsymbol{z}_0}\middle\|P^{(Q_{\boldsymbol{\Theta}_k},\lambda_k)}_{\boldsymbol{\Theta}_k|\boldsymbol{Z}_k=\boldsymbol{z}_k}\right)\mathrm{d}P_{\boldsymbol{Z}_0}(\boldsymbol{z}_0)\\
&=\int \mathrm{D}\left(P^{(m)}_{\boldsymbol{\Theta}_k}\middle\|P^{(Q_{\boldsymbol{\Theta}_k},\lambda_k)}_{\boldsymbol{\Theta}_k|\boldsymbol{Z}_k=\hat{\boldsymbol{z}}_k}\right)\mathrm{d}P_{\boldsymbol{Z}_k}(\hat{\boldsymbol{z}}_k)\\
&\quad+\int \mathrm{D}\left(P^{(m)}_{\boldsymbol{\Theta}_k|\boldsymbol{Z}_0=\boldsymbol{z}_0}\middle\|P^{(m)}_{\boldsymbol{\Theta}_k}\right)\mathrm{d}P_{\boldsymbol{Z}_0}(\boldsymbol{z}_0),
\end{aligned}
$$



Thank you for your attention!

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